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Characterizing Homotopy of Systems of Curves on a Compact Surface by Crossing Numbers

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ABSTRACT

Let $C_1, \dots, C_k$ and $C'_1, \dots, C'_k$ be closed curves on a compact surface S. We characterize (in terms of counting crossings) when there exists a permutation π of $\{1, \dots, k\}$ such that $C'_{\pi(i)}$ is freely homotopic to C_i or C_i^{-1}, for each $i = 1, \dots, k$.

1. INTRODUCTION

Let S denote a compact surface without boundary. A *closed curve* C on S is a continuous function $C: S^1 \to S$, where S^1 is the unit circle $\{z \in \mathbb{C}|$

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$|z| = 1\}$. Two closed curves C and C' are called *freely homotopic*, in notation $C \sim C'$, if there exists a continuous function $\Phi: [0,1] \times S^1 \to S$ such that $\Phi(0, z) = C(z)$ and $\Phi(1, z) = C'(z)$, for all $z \in S^1$.

Two systems of closed curves $C_1, \ldots, C_k$ and $C'_1, \ldots, C'_{k'}$, are called *homotopically equivalent* if $k = k'$ and there exists a permutation π of $\{1, \ldots, k\}$ such that, for each $i = 1, \ldots, k$, one has $C'_{\pi(i)} \sim C_i$ or $C'_{\pi(i)} \sim C_i^{-1}$.

In this paper we characterize homotopic equivalence of systems of curves in terms of minimum crossing numbers of curves. This generalizes the result of [6], where a characterization is given for compact *orientable* surfaces.

To describe the characterization, define for closed curves C and D,

$$\begin{aligned} \mathrm{cr}(C, D) &:= |\{(y, z) \in S^1 \times S^1 | C(y) = D(z)\}|, \\ \mathrm{mincr}(C, D) &:= \min\{\mathrm{cr}(C', D') | C' \sim C, D' \sim D\}. \end{aligned} \tag{1}$$

A closed curve C is called *orientation-preserving* if passing once through C does not change the meaning of "left" and "right." Otherwise, C is called *orientation-reversing*. C is called *orientation-primitive* if there do not exist an orientation-preserving curve D and an integer $n \geqslant 2$ so that $C \sim D^n$. [For a closed curve C and an integer n, C^n is the closed curve defined by $C^n(z) := C(z^n)$ for $z \in S^1$.] So each orientation-reversing closed curve is orientation-primitive.

We show the following theorem:

THEOREM 1. *Let $C_1, \ldots, C_k$ and $C'_1, \ldots, C'_{k'}$, be orientation-primitive closed curves on a compact surface S. Then the following are equivalent*:

(i) *$C_1, \ldots, C_k$ and $C'_1, \ldots, C'_{k'}$, are homotopically equivalent.*
(ii) *For each closed curve D on S,*

$$\sum_{i=1}^{k} \mathrm{mincr}(C_i, D) = \sum_{i=1}^{k'} \mathrm{mincr}(C'_i, D). \tag{2}$$

2. A LINEAR ALGEBRAIC FORMULATION

The theorem can be formulated equivalently as the nonsingularity of a certain infinite symmetric matrix. Let $\mathscr{C}$ be the family of free homotopy classes of closed curves on S. For $\Gamma, \Delta \in \mathscr{C}$, define $\mathrm{mincr}(\Gamma, \Delta) := \mathrm{mincr}(C, D)$ for (arbitrary) $C \in \Gamma$ and $D \in \Delta$. So mincr is considered here as a function from $\mathscr{C} \times \mathscr{C}$ to $\mathbb{Z}_+$. We can represent this function as an infinite symmetric matrix M with both rows and columns indexed by $\mathscr{C}$.

The rows of M are not linearly independent. First of all, the row corresponding to the trivial class $\langle 0 \rangle$ is all-zero (where 0 denotes a homotopically trivial closed curves and where $\langle \cdot\cdot \rangle$ denotes the equivalence class containing $\cdot\cdot$). Moreover, the rows corresponding to $\langle C \rangle$ and $\langle C^{-1} \rangle$ are the same, as $\text{mincr}(C, D) = \text{mincr}(C^{-1}, D)$ for each closed curve D. Moreover, it is shown in [7] that for each pair of orientation-preserving closed curves C, D and each $n \in \mathbb{Z}$ one has $\text{mincr}(C^n, D) = |n|\text{mincr}(C, D)$. In fact, this also holds if D is orientation-reversing, so the row corresponding to $\langle C^n \rangle$ is a multiple of the row corresponding to $\langle C \rangle$.

Now the theorem states that if we restrict ourselves to orientation-primitive closed curves, then the rows of M are linearly independent. To formulate this precisely, choose $\mathscr{C}' \subseteq \{\langle C \rangle | C \text{ orientation-primitive}\}$ such that for each orientation-primitive closed curve, exactly one of $\langle C \rangle$ and $\langle C^{-1} \rangle$ belongs to $\mathscr{C}'$. Let M' be the $\mathscr{C}' \times \mathscr{C}'$ submatrix of M. Then the following theorem is equivalent to the theorem above:

THEOREM 2. *The matrix M' is nonsingular, i.e., the rows of M' are linearly independent.*

Proof. The proof is similar to that in [6]. ■

3. CLOSED CURVES IN GRAPHS

Let $G = (V, E)$ be an undirected graph, without loops and parallel edges, embedded on a compact surface S and where each vertex of G has degree 2 or 4. Let W be the set of vertices of degree 4. For each vertex $v \in W$, we can order the edges incident with v cyclically. For each $v \in W$, we fix one such ordering $e_1^v, e_2^v, e_3^v, e_4^v$. We say that e_1^v and e_3^v are *opposite in* v, and similarly for e_2^v and e_4^v.

We identify G with its embedding on S. (An edge is considered as an open line segment.) So we can speak of a closed curve C in G, which is a continuous function $C: S^1 \to G$. We say that C is *nonreturning* if $C|K$ is one-to-one, for each edge e of G and each component K of $C^{-1}(\bar{e})$. (Here $\bar{e}$ is the closure of e.)

We say that C is *straight* if C is nonreturning and in each vertex $v \in W$, if C arrives in v over an edge e, it leaves v over the edge opposite in v to e.

A *straight decomposition* of G is a collection of straight closed curves such that each edge is traversed exactly once. Such a straight decomposition is unique up to a number of trivial operations.

Let C be a closed curve in G. For any edge e of G, we define

$$\mathrm{tr}_C(e) := \text{number of times } C \text{ traverses } e. \tag{3}$$

[More precisely, it is the number of components of $C^{-1}(e)$.] For any vertex of degree 4 in G, we define

$$\alpha_{ij}^v(C) := \begin{array}{l}\text{number of times } C \text{ traverses } v\\ \text{by going from } e_i^v \text{ to } e_j^v \text{ or from } e_i^v.\end{array} \tag{4}$$

The following two propositions generalize Lemma A in [6], and the proofs are similar (note that Lemmas A and B in [6] do not use the orientability of the surface).

We define for any closed curve C on a surface S,

$$\begin{aligned}\mathrm{cr}(C) &:= \tfrac{1}{2}|\{(y,z) \in S^1 \times S^1 | C(y) = C(z) \text{ and } y \neq z\}|,\\ \mathrm{mincr}(C) &:= \min\{\mathrm{cr}(C') | C' \sim C\}.\end{aligned} \tag{5}$$

PROPOSITION 1. *For any nonreturning closed curve C in G,*

$$\mathrm{mincr}(C) \leqslant \sum_{v \in W} \left[\alpha_{13}^v(C)\alpha_{24}^v(C) + \frac{1}{4} \sum_{1 \leqslant g < h \leqslant 4} \sum_{\substack{1 \leqslant k < l \leqslant 4\\ |\{g,h\} \cap \{k,l\}| = 1}} \alpha_{gh}^v(C)\alpha_{kl}^v(C) \right]. \tag{6}$$

PROPOSITION 2. *For any pair of nonreturning closed curves C, D in G with $C \neq D$,*

$$\mathrm{mincr}(C,D) \leqslant \sum_{v \in W} \left[\alpha_{13}^v(C)\alpha_{24}^v(D) + \alpha_{24}^v(C)\alpha_{13}^v(D) + \frac{1}{2} \sum_{1 \leqslant g < h \leqslant 4} \sum_{\substack{1 \leqslant k < l \leqslant 4\\ |\{g,h\} \cap \{k,l\}| = 1}} \alpha_{gh}^v(C)\alpha_{kl}^v(D) \right]. \tag{7}$$

If $C_1, \ldots, C_s$ are edge-disjoint closed curves in G, then clearly $|W| \geqslant \sum_{i=1}^{s} \operatorname{mincr}(C_i) + \sum_{i<j} \operatorname{mincr}(C_i, C_j)$. The next proposition gives a lower bound for $|W|$ in case the closed curves $C_1, \ldots, C_s$ are "fractionally" edge-disjoint as described in (8) below.

PROPOSITION 3. *Let $C_1, \ldots, C_s$ be nonreturning closed curves in G and let $\lambda_1, \ldots, \lambda_s > 0$ be such that*

$$\sum_{j=1}^{s} \lambda_j \operatorname{tr}_{C_j}(e) \leqslant 1, \quad \text{for each } e \in E. \tag{8}$$

Then

$$\sum_{i=1}^{s} \lambda_i^2 \operatorname{mincr}(C_i) + \sum_{\substack{i,j=1 \\ i<j}}^{s} \lambda_i \lambda_j \operatorname{mincr}(C_i, C_j) \leqslant |W|. \tag{9}$$

Equality in (9) *implies that each C_i is straight.*

Proof. By Propositions 1 and 2, we obtain

$$\sum_{i=1}^{s} 2\lambda_i^2 \operatorname{mincr}(C_i) + \sum_{\substack{i,j=1 \\ i\neq j}}^{s} \lambda_i \lambda_j \operatorname{mincr}(C_i, C_j)$$

$$\leqslant \sum_{v \in W} \sum_{i,j=1}^{s} \lambda_i \lambda_j \Bigg[\alpha_{13}^{v}(C_i) \alpha_{24}^{v}(C_j) + \alpha_{24}^{v}(C_i) \alpha_{13}^{v}(C_j)$$

$$+ \frac{1}{2} \sum_{g<h} \sum_{\substack{k<l \\ |\{g,h\} \cap \{k,l\}| = 1}} \alpha_{gh}^{v}(C_i) \alpha_{kl}^{v}(C_j) \Bigg]. \tag{10}$$

For any vertex $v \in W$ and $g, h \in \{1, 2, 3, 4\}$, define

$$\alpha_{gh}^{v} := \sum_{j=1}^{s} \lambda_j \alpha_{gh}^{v}(C_j). \tag{11}$$

The right-hand side of (10) is equal to

$$\sum_{v\in W}\left[2\alpha_{13}^{v}\alpha_{24}^{v}+\tfrac{1}{2}\sum_{g<h}\sum_{\substack{k<l\\ |\{g,h\}\cap\{k,l\}|=1}}\alpha_{gh}^{v}\alpha_{kl}^{v}\right],$$

so it is sufficient to show that for any fixed vertex $v\in W$,

$$2\alpha_{13}^{v}\alpha_{24}^{v}+\tfrac{1}{2}\sum_{g<h}\sum_{\substack{k<l\\ |\{g,h\}\cap\{k,l\}|=1}}\alpha_{gh}^{v}\alpha_{kl}^{v}\leqslant 2. \tag{12}$$

This follows from Lemma B in [6], which lemma also implies that equality in (12) is attained only if $\alpha_{13}^{v}=\alpha_{24}^{v}=1$ and $\alpha_{12}^{v}=\alpha_{14}^{v}=\alpha_{23}^{v}=\alpha_{34}^{v}=0$. This shows the proposition. ■

4. CROSSINGS OF CLOSED CURVES ON SURFACES

We need a few observations on crossing numbers on surfaces, for which we make use of formulas given in [3], expressing $\operatorname{mincr}(C)$ and $\operatorname{mincr}(C,D)$ in $\operatorname{mincr}(J)$ and $\operatorname{mincr}(J,K)$, where J and K are geodesic; that is, J is a closed curve for which $C\sim J^{n}$ for some $n\geqslant 1$ and such that J is shortest with respect to a euclidean or hyperbolic distance on the surface (cf. [4]). First we have the following proposition:

PROPOSITION 4. *Let C be an orientation-reversing closed curve on S. Then* $\operatorname{mincr}(C,C^{2})<2\operatorname{mincr}(C,C)$.

Proof. Let J be the geodesic such that $C\sim J^{n}$, for some $n\in\mathbb{N}$. So J is orientation-reversing and n is odd. Then $\operatorname{mincr}(C,C)=2n^{2}\operatorname{mincr}(J)+n$ and $\operatorname{mincr}(C,C^{2})=4n^{2}\operatorname{mincr}(J)$. ■

Moreover:

PROPOSITION 5. *Let C and D be closed curves on S. Then* $\operatorname{mincr}(C,D^{2})\leqslant 2\operatorname{mincr}(C,D)$.

Proof. Choose C, D such that $\mathrm{cr}(C, D) = \mathrm{mincr}(C, D)$. Then $\mathrm{mincr}(C, D^2) \leqslant \mathrm{cr}(C, D^2) = 2\,\mathrm{cr}(C, D) = 2\,\mathrm{mincr}(C, D)$. ■

For a closed curve C on S, let $\mathrm{odd}(C) := 1$ if C is orientation-reversing, and $\mathrm{odd}(C) := 0$ if C is orientation-preserving.

PROPOSITION 6. *Let C be an orientation-primitive closed curve on S. Then* $\mathrm{mincr}(C, C) = 2\,\mathrm{mincr}(C) + \mathrm{odd}(C)$.

Proof. Let J be a geodesic such that $C \sim J^n$ for some $n \in \mathbb{N}$. If C is orientation-reversing, then J is orientation-reversing and n is odd, and hence $\mathrm{mincr}(C, C) = 2n^2\,\mathrm{mincr}(J) + n = 2\,\mathrm{mincr}(C) = 1$. If C and J are orientation-preserving, then $n = 1$ (as C is orientation-primitive), and hence $\mathrm{mincr}(C, C) = 2n^2\,\mathrm{mincr}(J) = 2\,\mathrm{mincr}(C)$. If C is orientation-preserving and J is orientation-reversing, then $n = 2$, and hence $\mathrm{mincr}(C, C) = 2n^2\,\mathrm{mincr}(J) = 2\,\mathrm{mincr}(C)$. ■

5. PROOF OF THEOREM 1

The implication (i) ⇒ (ii) in Theorem 1 is trivial as $\mathrm{mincr}(C^{-1}, D) = \mathrm{mincr}(C, D)$ for any pair of closed curves C, D on S. We show (ii) ⇒ (i).

Suppose by contradiction that $C_1, \ldots, C_k$ and $C'_1, \ldots, C'_{k'}$ are two systems of curves satisfying (ii) but not (i) such that $k + k'$ is minimal. This implies that:

$$\text{there are no } i \in \{1, \ldots, k\} \text{ and } j \in \{1, \ldots, k'\} \text{ such that}$$
$$C_i \sim C'_j \text{ or } C_i^{-1} \sim C'_j. \tag{13}$$

By symmetry we may assume that

$$\sum_{i=1}^{k'} \mathrm{mincr}(C'_i) + \sum_{\substack{i,j=1 \\ i<j}}^{k'} \mathrm{mincr}(C'_i, C'_j) \leqslant \sum_{i=1}^{k} \mathrm{mincr}(C_i) + \sum_{\substack{i,j=1 \\ i<j}}^{k} \mathrm{mincr}(C_i, C_j). \tag{14}$$

It is a basic fact (cf. [1, 5, 8]), that there exist $\tilde{C}_1 \sim C'_1, \dots, \tilde{C}_{k'} \sim C'_{k'}$ such that

$$\begin{aligned} \operatorname{cr}\left(\tilde{C}_i\right) &= \operatorname{mincr}(C'_i), \quad \text{for } i = 1, \dots, k', \\ \operatorname{cr}\left(\tilde{C}_i, \tilde{C}_j\right) &= \operatorname{mincr}(C'_i, C'_j), \quad \text{for } i, j = 1, \dots, k' \text{ and } i \neq j \end{aligned} \tag{15}$$

The result being invariant under homotopies, we may assume that $\tilde{C}_i = C'_i$, for $i = 1, \dots, k'$, and that each point of S is traversed at most twice by the C'_i (so no two crossings of the C'_i coincide).

Let $G = (V, E)$ be the graph made up by the curves C'_i. So G is a graph embedded on S. Each point of S traversed twice by the C'_i is a vertex of degree 4 of G. Moreover, we take as vertices some of the points of S traversed exactly once by the C'_i, in such a way that G will be a graph without loops or parallel edges. So each vertex of G has degree 2 or 4 and $C'_1, \dots, C'_{k'}$, is a straight decomposition of G. Let W denote the set of vertices of degree 4. We obtain:

$$|W| = \sum_{i=1}^{k'} \operatorname{mincr}(C'_i) + \sum_{\substack{i,j=1 \\ i<j}}^{k'} \operatorname{mincr}(C'_i, C'_j). \tag{16}$$

By (2) for each closed curve $D: S^1 \to S \setminus V$,

$$\operatorname{cr}(G, D) = \sum_{i=1}^{k'} \operatorname{cr}(C'_i, D) \geqslant \sum_{i=1}^{k'} \operatorname{mincr}(C'_i, D) = \sum_{i=1}^{k} \operatorname{mincr}(C_i, D), \tag{17}$$

where $\operatorname{cr}(G, D) := |\{z \in S^1 | D(z) \in G\}|$. Hence, by the "homotopic circulation theorem" in [2], there exist closed curves $D_1, \dots, D_s$, with rationals $\lambda_1, \dots, \lambda_s > 0$ and a partition $S_1, \dots, S_k$ of $\{1, \dots, s\}$ such that

$$\begin{aligned} D_j &\sim C_i, \quad \text{for } i = 1, \dots, k \text{ and } j \in S_i, \\ \sum_{j \in S_i} \lambda_j &= 1, \quad \text{for } i = 1, \dots, k, \\ \sum_{j=1}^{s} \lambda_j \operatorname{tr}_{D_j}(e) &\leqslant 1, \quad \text{for } e \in E. \end{aligned} \tag{18}$$

Clearly, we may assume the D_j to be nonreturning. This implies with Propositions 3 and 6,

$$2\sum_{i=1}^{k}\operatorname{mincr}(C_i)+\sum_{\substack{i,j=1\\ i\neq j}}^{k}\operatorname{mincr}(C_i,C_j)$$

$$=\sum_{i,j=1}^{k}\operatorname{mincr}(C_i,C_j)-\sum_{i=1}^{k}\operatorname{odd}(C_i)$$

$$=\sum_{g,h=1}^{s}\lambda_g\lambda_h\operatorname{mincr}(D_g,D_h)-\sum_{i=1}^{k}\operatorname{odd}(C_i)$$

$$=\sum_{\substack{g,h=1\\ h\neq g}}^{s}\lambda_g\lambda_h\operatorname{mincr}(D_g,D_h)+\sum_{g=1}^{s}\lambda_g^2\operatorname{mincr}(D_g,D_g)$$

$$-\sum_{i=1}^{k}\operatorname{odd}(C_i)$$

$$=\sum_{\substack{g,h=1\\ h\neq g}}^{s}\lambda_g\lambda_h\operatorname{mincr}(D_g,D_h)+\sum_{g=1}^{s}\lambda_g^2\left(2\operatorname{mincr}(D_g)+\operatorname{odd}(D_g)\right)$$

$$-\sum_{i=1}^{k}\operatorname{odd}(C_i)$$

$$\leqslant 2|W|+\sum_{i=1}^{k}\operatorname{odd}(C_i)\left(-1+\sum_{g\in S_i}\lambda_g^2\right)\leqslant 2|W|. \tag{19}$$

(The first inequality follows from Proposition 3.)

By our assumption (14) and by (16), we should have equality throughout in (19). Hence by Proposition 3, each curve D_j ($j=1,\dots,s$) is straight. So there exists a function $\pi:\{1,\dots,s\}\to\{1,\dots,k'\}$ and $n_1,\dots,n_s$ such that

$$D_j=C'^{n_j}_{\pi(j)}\quad\text{or}\quad D_j=C'^{-n_j}_{\pi(j)},\quad\text{for } j=1,\dots,s. \tag{20}$$

For each $j=1,\dots,s$, by (13), $n_j\geqslant 2$, and, as each C_i is orientation-primitive, $C'_{\pi(j)}$ is orientation-reversing.

Suppose that C_i is orientation-reversing for some $i \in \{1, \dots, k\}$. It follows from

$$\sum_{i=1}^{k} \operatorname{odd}(C_i)\left(-1 + \sum_{g \in S_i} \lambda_g^2\right) = 0 \tag{21}$$

that $|S_i| = 1$, say $S_i = \{j\}$. We now obtain $\lambda_j = 1$ and $D_j = C_i'$ or $D_j = C_i'^{-1}$, contradicting (13). Hence C_i is orientation-preserving for $i = 1, \dots, k$.

So for $j = 1, \dots, k'$ we have that n_j is even and, hence, as C_i is orientation-primitive, $n_j = 2$ and $C'_{\pi(j)}$ is orientation-reversing for $j = 1, \dots, s$. Hence, using Propositions 4 and 5, and assuming without loss of generality that $\pi(1) = 1$,

$$\begin{aligned}\sum_{i=1}^{k} \operatorname{mincr}(C_i, C_1') &= \sum_{j=1}^{s} \lambda_j \operatorname{mincr}(D_j, C_1') = \sum_{j=1}^{s} \lambda_j \operatorname{mincr}\left(C'^2_{\pi(j)}, C_1'\right) \\ &< \sum_{j=1}^{s} 2\lambda_j \operatorname{mincr}(C'_{\pi(j)}, C_1') \leqslant \sum_{i=1}^{k'} \operatorname{mincr}(C_i', C_1'). \end{aligned} \tag{22}$$

Here the last inequality follows from the fact that, for any $i = 1, \dots, k$, the sum of those λ_j for which $\pi(j) = i$ is at most $\frac{1}{2}$, by (8). However, (22) contradicts (2). ∎

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